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# Stationary axially-symmetric electrovac fields with reflectional symmetry

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**Abstract.** The Einstein–Maxwell equations for axially-symmetric stationary fields are investigated. The existence of a class of solutions having reflectional symmetry is predicted and examples are given.

## 1. The field equations and introduction

We study the Einstein–Maxwell equations for axially-symmetric stationary systems in the two forms given by Bonnor (1973) and Ernst (1968). We write the metric in the form

$$ds^2 = -e^\lambda (dz^2 + dr^2) - f^{-1} r^2 \, d\alpha^2 + f(dt - w \, d\alpha)^2. \tag{1}$$

All functions in this paper depend on $z$ and $r$ only. The electric and magnetic potentials are denoted by $\phi \equiv A_4$ and $\psi \equiv A_3'$, respectively. The four functions $\psi, \phi, w, f$ are to be found either from the four equations

$$\nabla^2 \psi = f^{-1}(f_1 \psi_1 + f_2 \psi_2) + r^{-1} f(w_2 \phi_1 - w_1 \phi_2) \tag{2a}$$

$$\nabla^2 \phi = f^{-1}(f_1 \phi_1 + f_2 \phi_2) + r^{-1} f(w_1 \psi_2 - w_2 \psi_1) \tag{2b}$$

$$\nabla^2 w - 2r^{-1} w_2 = -2f^{-1}(f_1 w_1 + f_2 w_2) + 4rf^{-2}(\psi_1 \phi_2 - \psi_2 \phi_1) \tag{2c}$$

$$\nabla^2 f - f^{-1}(f_1^2 + f_2^2) = 2(\phi_1^2 + \phi_2^2 + \psi_1^2 + \psi_2^2) - r^{-2} f^3 (w_1^2 + w_2^2) \tag{2d}$$

where

$$(z, r, \alpha, t) \equiv (x_1, x_2, x_3, x_4) \qquad \text{and} \qquad \nabla^2 \equiv \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right),$$

or from the two complex Ernst equations

$$(\mathrm{Re}\, E + |F|^2)\nabla^2 F = (\nabla E + 2F^* \nabla F) \cdot \nabla F \tag{3a,b}$$

$$(\mathrm{Re}\, E + |F|^2)\nabla^2 E = (\nabla E + 2F^* \nabla F) \cdot \nabla E \tag{3c,d}$$

where

$$\nabla \equiv \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial r}\right), \qquad F \equiv \phi + i\psi \qquad (i = \sqrt{-1})$$

and $E$ is another complex function, linked with $\psi$, $\phi$, $w$, $f$. After (2) or (3) are solved, $\lambda$ is obtained from

$$\lambda_1 = -f^{-1}f_1 + rf^{-2}f_1f_2 - 4rf^{-1}(\phi_1\phi_2 + \psi_1\psi_2) - r^{-1}f^2 w_1 w_2 \tag{4a}$$

$$\lambda_2 = -f^{-1}f_2 + \tfrac{1}{2}rf^{-2}(f_2^2 - f_1^2) + 2rf^{-1}(\phi_1^2 - \phi_2^2 + \psi_1^2 - \psi_2^2) + \tfrac{1}{2}r^{-1}f^2(w_1^2 - w_2^2). \tag{4b}$$

In order to find solutions of (2) or (3), it is customary to make simplifying assumptions. To assume, for instance, that the four functions occurring in (3) are functions of $r^2 + nz^2$ reduces (3) to a system of ordinary differential equations if and only if $n = 0$, 1 or $-2$. In this paper we make assumptions which reduce the number of equations and unknown functions from four to three.

## 2. Simplifying assumptions

*Theorem A.* If the equations

$$w(z, r) = w(-z, r) \tag{5a}$$

$$f(z, r) = f(-z, r) \tag{5b}$$

$$\phi(z, r) = \psi(-z, r) \qquad (\Rightarrow \psi(z, r) = \phi(-z, r)) \tag{5c}$$

are satisfied, and if (2$a$) is satisfied at $(z_1, r_1, \alpha_1, t_1)$ then (2$b$) is satisfied at $(-z_1, \alpha_1, r_1, t_1)$. Equation (5$c$) allows us, at least in principle, to express $\phi$ in terms of $\psi$ and we have thus three equations (2$a$,$c$,$d$) for the three unknowns $\psi$, $w$, $f$. The proof is simple. By (5$a$) we have that $w$ is an even function of $z$, hence $w_1$ is odd and $w_2$ is even in $z$; and the same applies to $f$. Similarly

$$\phi(z, r) = \psi(-z, r) \Rightarrow \qquad \phi_1(z, r) = -\psi_1(-z, r),\ \phi_2(z, r) = \psi_2(-z, r), \text{etc.}$$

Calculating (2$a$) for $z = z_1$ gives (2$b$) for $z = -z_1$. Writing out the real (3$a$) and imaginary (3$b$) part of the Maxwell equations (3$a$,$b$), it is also easy to prove

*Theorem B.* If the equations

$$\operatorname{Re} E(z, r) = \operatorname{Re} E(-z, r) \tag{6a}$$

$$\operatorname{Im} E(z, r) = -\operatorname{Im}(-z, r) \tag{6b}$$

$$\phi(z, r) = \psi(-z, r) \tag{6c}$$

are satisfied and if (3$a$) is satisfied at $(z_1, r_1, \alpha_1, t_1)$ then (3$b$) is satisfied at $(-z_1, r_1, \alpha_1, t_1)$. We could show in this fashion

*Theorem C.* If the equations

$$f(z, r) = f(z, -r), \qquad w(z, r) = -w(z, -r), \qquad \phi(z, r) = \psi(z, -r) \tag{7}$$

are satisfied and if (2$a$) is satisfied at $(z_1, r_1, \alpha_1, t_1)$ then (2$b$) is satisfied at $(z_1, -r_1, \alpha_1, t_1)$.

*Theorem D.* If the equations

$$\text{Re}\,E(z,r)=\text{Re}\,E(z,-r),\qquad \text{Im}\,E(z,r)=-\text{Im}\,E(z,-r),\qquad \phi(z,r)=\psi(z,-r) \tag{8}$$

are satisfied and if (3$a$) is satisfied at $(z_1, r_1, \alpha_1, t_1)$ then (3$b$) is satisfied at $(z_1, -r_1, \alpha_1, t_1)$.

## 3. Examples

Guided by the above theorems, some simple solutions have been found. They are displayed in table 1 ($a$, $b$, $c$ are real constants).

Solutions II$a$,$b$,$c$ are linked by duality rotations. To generalize II, leave $\psi$, $\phi$ unchanged but replace $f$, $w$ by

$$f=r/w=r(8c^2r+a+b\ln r).$$

Solutions II and III bear some resemblance to those given by Arbex and Som (1973), however their $\lambda$ (their equation (2.23)×2) does not seem to satisfy their equation ((2.11)+(2.12)).

## 4. Reflectional symmetry

Michalski and Wainwright (1975) demonstrate that invariance of the metric $g_{ik}$, $(i, k=1, 2, 3, 4)$ under a continuous coordinate transformation does not always imply the same invariance for the electromagnetic field tensor $F_{ik}$. We now investigate in the same spirit the invariance of solutions satisfying (5) under the discrete transformation

$$z'=-z,\qquad r'=r,\qquad \alpha'=\alpha,\qquad t'=t. \tag{9}$$

Specializing the definition of invariant tensors (Florides *et al* 1965, § 2) to (9) we define any tensor $T_{ik}$ (or $T^{ik}$) to be invariant under the reflection (9) if

$$T_{11}(z,r)=+T_{11}(-z,r),\qquad T_{1\nu}(z,r)=-T_{1\nu}(-z,r),$$
$$T_{\mu\nu}(z,r)=+T_{\mu\nu}(-z,r), \tag{10}$$

where $\mu, \nu=2, 3, 4$.

From (1) we find the nonzero $g_{ik}$ to be

$$g_{11}=g_{22}=-e^{\lambda},\qquad g_{33}=fw^2-r^2f^{-1},\qquad g_{34}=-fw,\qquad g_{44}=fw^2. \tag{11}$$

By using (5) and (4) we find $\lambda$, and thus $e^{\lambda}$, to be even in $z$; this together with (5$a$, $b$, 10, 11) shows that any metric satisfying (5) has reflectional symmetry.

For the investigation of the symmetry of $F_{ik}$ we need (Bonnor 1973)

$$F_{4a}=\phi_{,a} \tag{12a}$$

$$F^{ab}=(fe^{\lambda}r)^{-1}\epsilon^{abc}\psi_{,c} \tag{12b}$$

where $a, b, c=1, 2, 3$; and $\epsilon^{abc}$ is the permutation symbol with values $\pm 1$ and 0. It follows from (12$b$) that $\psi$ cannot be a scalar (e.g. $z'=-z \Rightarrow \psi'(z')=-\psi(z)$).

*Theorem E.* For solutions satisfying (5), $F_{ik}$ does not have reflectional symmetry;

**Table 1.**

| Solution | Example to theorem | $\psi$ | $\phi$ | $w$ | $f$ | $\lambda$ | Re $E$ | Im $E$ |
|---|---|---|---|---|---|---|---|---|
| I | $A,B,C,D$ | $a(r^2+z^2)^{1/2}$ | $a(r^2+z^2)^{1/2}$ | $0$ | $-2a^2z^2$ | $-2\ln\|(r^2/z)+z\|+i\pi$ | $-2a^2(r^2+2z^2)$ | $0$ |
| II$a$ | $A,B$ | $c(r-z)$ | $c(r+z)$ | $(8c^2r)^{-1}$ | $8c^2r^2$ | $-\frac{1}{2}\ln r$ | $2c^2(3r^2-z^2)$ | $-4c^2rz$ |
| II$b$ | $C,D$ | $c(-r-z)$ | $c(r-z)$ | $(8c^2r)^{-1}$ | $8c^2r^2$ | $-\frac{1}{2}\ln r$ | $2c^2(3r^2-z^2)$ | $-4c^2rz$ |
| II$c$ | $E$ | $-\sqrt{2}cz$ | $\sqrt{2}cr$ | $(8c^2r)^{-1}$ | $8c^2r^2$ | $-\frac{1}{2}\ln r$ | $2c^2(3r^2-z^2)$ | $-4c^2rz$ |
| III | $A,B$ | $\frac{3}{2}(rb)^{2/3}$ | $\frac{3}{2}(rb)^{2/3}$ | $3r^{2/3}b^{-1/3}$ | $(rb)^{2/3}$ | $-9(rb)^{2/3}-\frac{4}{9}\ln r$ | $(rb)^{2/3}-\frac{9}{2}(rb)^{4/3}$ | $2bz$ |

however the duality rotation

$$\bar{\psi} = 2^{-1/2}(\psi - \phi) \tag{13a}$$

$$\bar{\phi} = 2^{-1/2}(\psi + \phi) \tag{13b}$$

results in a new $F_{ik}$ (denoted by $\bar{F}_{ik}$) having this symmetry. We prove the second part of this theorem. Using (13*b*) and (5*c*) we have

$$\sqrt{2}\bar{\phi} = \phi(z, r) + \psi(z, r) = \phi(z, r) + \phi(-z, r). \tag{14a}$$

This is even in $z$ and we find with (12*a*) as required by (10) that $F_{41}$ is odd in $z$ and $F_{42}$ is even. By (13*a*) and (5*c*) we have

$$\sqrt{2}\bar{\psi} = \psi(z, r) - \phi(z, r) = \phi(-z, r) - \phi(z, r) \tag{14b}$$

is odd in $z$. Hence, using (12*b*), (5*a*) and the fact that $\lambda$ is even gives

$$F^{23} = (\text{a function even in } z) \times \bar{\psi}_{,1} \qquad \text{is even in } z$$

and

$$F^{13} = (\text{a function even in } z) \times \bar{\psi}_{,2} \qquad \text{is odd in } z,$$

as required by (10). From this and the reflectional symmetry of $g_{ik}$ we find that the remaining nonzero $F_{ik}$ (i.e. $F_{13}$ and $F_{23}$) satisfy (10) as well. This concludes the proof. For example, $F_{ik}$ of solution II*c*, which was obtained from II*a* by (13), has reflectional symmetry.

## 5. Summary and possible physical significance

In order to reduce the number of independent equations in (2) or (3) by one, we imposed conditions (5) or (6). To apply (5*c*) or (6*c*), which assumes that $\phi$ is the mirror image of $\psi$, in approximation methods, should be easy; if we expand $\phi$ as a Taylor series at $z = 0$, the series for $\psi$ is then immediately known. How to apply (5*c*) in the search for exact solutions is less obvious, but its usefulness in conjunction with other simplifying assumptions has been demonstrated by the ease with which the solutions of § 3 have been obtained. In § 4 we showed that a $g_{ik}$ satisfying (5) or (6) has reflectional symmetry and that there exists a duality rotation which yields an $F_{ik}$ with this symmetry.

It is well known in special relativity that the Maxwell tensor for a magnetic and an electric monopole at rest (say at $(z, r) = (d, 0)$ and $(-d, 0)$ respectively) corresponds to electromagnetic energy rotating about the $z$ axis. We therefore expect that such a system in general relativity corresponds to a solution of (2) satisfying (5) with $w \neq 0$. If we could find such solutions we would know under which circumstances electric and magnetic monopoles coexist. This might provide us with clues in the search for magnetic monopole particles.

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